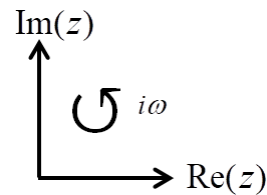
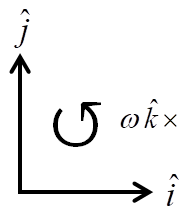


Here is another demonstration that $i = \hat{k} \times$ is the complex number. I sat engineering mechanics some time ago, and all the way through it I was using $i = \hat{k} \times$ to solve the problems. Probably more than 90% of the entire course content (kinematics+energy+vibration) explicitly involved this, particularly kinematics in which every problem involved $i = \hat{k} \times$. The remaining 10% of the course content involved $i = \hat{k} \times$ implicitly when not explicitly involved in the calculations (in vibration for example). Whilst studying for the final exam, I was reworking the course problems and did some of them on the R^2 vector plane with $\hat{k} \times$, and then again on the (notional) Argand diagram with i (that is, with complex number notation). The results are exactly the same, one-for-one, with $i = \hat{k} \times$ as the complex number. This demonstrates exact equivalence between (1) the Argand diagram of the complex plane and (2) the xy plane, generated by $\hat{i}$ and $\hat{j}$, with $\hat{k}$ as the implicit z-axis (and rotation operator when appended with $\times$). So here are two examples from engineering mechanics, in particular, relative motion kinematics. The first example is a **relative motion velocity problem** and the second is a **relative motion acceleration problem**, inclusive of the Coriolis term.

Note that for rotations, the operator is $\bar{\omega} \times$. Because it is plane kinematics (kinematic motion restricted to R^2), this is always a single component cross product and so equal to $\bar{\omega} \times = \omega \hat{k} \times$ (giving it the exact properties of the complex number). So, on the Argand plane, this is $\bar{\omega} \times = \omega k \times = \omega i$ where ω is the scalar angular velocity. The relevant algebraic and vector quantities are

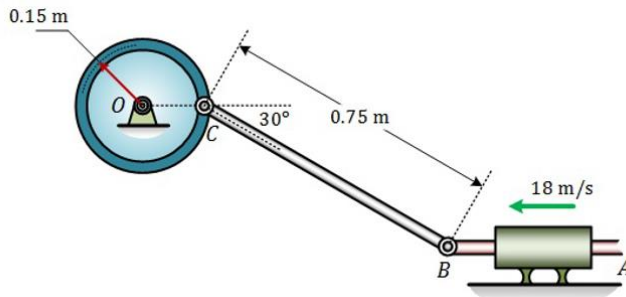


Position vector:	$\vec{r} = \hat{i}x + \hat{j}y$	$z = x + iy$
Even component:	$\hat{i}'\vec{r} = x$	$\text{Re}(z) = x$
Odd component:	$\hat{j}'\vec{r} = y$	$\text{Im}(z) = y$
90° Rotation operator:	$\hat{k} \times$	i
Reflection:	$(\hat{k} \times)^2 = -1$	$i^2 = -1$
Velocity vector:	$\dot{\vec{r}} = \hat{i}\dot{x} + \hat{j}\dot{y}$	$\dot{z} = \dot{x} + i\dot{y}$
Acceleration vector:	$\ddot{\vec{r}} = \hat{i}\ddot{x} + \hat{j}\ddot{y}$	$\ddot{z} = \ddot{x} + i\ddot{y}$

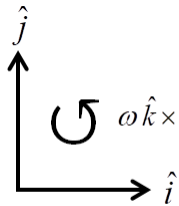
So there is no such thing as the complex field $\mathbb{C}$, since it is just $\mathbb{R}^2$ as a subset of $\mathbb{R}^3$, although complex algebra is still a very useful mathematical shorthand given the commutation properties of $i = \hat{k} \times$ and its deceptive appearance [before you know about it] as an algebraic scalar (when it is actually an $\mathbb{R}^3$ vector rotation operator [rather than a scalar or a number] that give it its ‘complex’ properties of $\sqrt{-1}$).

Example 1: Relative motion, velocity

(MMAN2300) If rod AB slides along the horizontal slot with a velocity of 18m/s, determine the angular velocities of link BC and the disk at the instant shown.



Solution by xy-plane vector algebra



Point B is moving to the left at 18 m/s

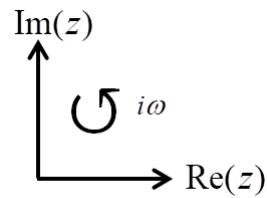
$$\vec{v}_B = \hat{i}(-18) + \hat{j}0 \text{ m/s} \quad (\text{data})$$

Velocity of point C, relative to O with O as a fixed point

$$\begin{aligned} \vec{v}_C &= \vec{v}_O + \vec{v}_{C/O} \\ &= \vec{v}_O + (\vec{\omega}_{OC} \times) \vec{r}_{C/O} \\ &= \vec{v}_O + (\omega_{OC} \hat{k} \times) (\vec{r}_C - \vec{r}_O) \\ &= \vec{0} + \omega_{OC} (\hat{k} \times) \hat{i} (0.15) \\ &= (\hat{k} \times \hat{i}) (0.15 \omega_{OC}) \\ &= \hat{j} (0.15 \omega_{OC}) \end{aligned}$$

C is moving in a vertical direction at the instant shown, since it is a pure $\hat{j}$ -hat component, as implied by the data geometry.

Solution by complex plane algebra



$$\dot{z}_B = -18 + 0i \text{ m/s} \quad (\text{data})$$

$$\begin{aligned} \dot{z}_C &= \dot{z}_O + \dot{z}_{C/O} \\ &= (0 + i0) + (\omega_{OC} i)(0.15 + i0) \\ &= 0.15 \omega_{OC} i \end{aligned}$$

which is purely imaginary nb: this can also be written as

$$\begin{aligned} \dot{z}_C &= \dot{z}_O + \dot{z}_{C/O} \\ &= (0 + 0\hat{k} \times) + (\omega_{OC} \hat{k} \times (0.15 + 0\hat{k} \times)) \\ &= 0.15 \omega_{OC} \hat{k} \times \end{aligned}$$

Velocity of point B, relative to C

$$\begin{aligned}
 \vec{v}_B &= \vec{v}_C + \vec{v}_{B/C} \\
 &= \vec{v}_C + (\vec{\omega}_{BC} \times) \vec{r}_{B/C} \\
 &= \vec{v}_C + (\omega_{BC} \hat{k} \times) (\vec{r}_B - \vec{r}_C) \\
 &= \vec{v}_C + (\omega_{BC} \hat{k} \times) (\hat{i}(0.75 \cos 30) + \hat{j}(-0.75 \sin 30)) \\
 &= \hat{j}0.15\omega_{OC} + (\hat{k} \times \hat{j})(\omega_{BC} 0.75 \cos 30) \\
 &\quad + (\hat{k} \times \hat{j})(-\omega_{BC} 0.75 \sin 30) \\
 &= \hat{j}0.15\omega_{OC} + \hat{j}(\omega_{BC} 0.75 \cos 30) + (-\hat{i})(-\omega_{BC} 0.75 \sin 30) \\
 &= \hat{i}(\omega_{BC} 0.75 \sin 30) + \hat{j}(0.15\omega_{OC} + \omega_{BC} 0.75 \cos 30)
 \end{aligned}$$

With $\vec{v}_B$ given as -18 m/s to the left $\vec{v}_B = \hat{i}(-18)$ this is then

$$\begin{aligned}
 \hat{i}(-18) + \hat{j}0 &= \hat{i}(\omega_{BC} 0.75 \sin 30) \\
 &\quad + \hat{j}(0.15\omega_{OC} + \omega_{BC} 0.75 \cos 30)
 \end{aligned}$$

Solving for the unknown angular velocities from the i -hat and j -hat components

$$-18 = \omega_{BC} 0.75 \sin 30$$

$$0 = 0.15\omega_{OC} + \omega_{BC} 0.75 \cos 30$$

which solve for

$$\begin{aligned}
 \omega_{BC} &= -18 / 0.75 \sin 30 \\
 &= -48 \text{ rad/s (CW)}
 \end{aligned}$$

$$\begin{aligned}
 \omega_{OC} &= -\omega_{BC} 0.75 \cos 30 / 0.15 \\
 &= 48(0.75 \cos 30 / 0.15) \\
 &= 207.85 \text{ rad/s (CCW)}
 \end{aligned}$$

Also, since the rotation of OC is counter-clockwise, the velocity of C is straight up (the j -hat direction), at that instant, with velocity vector

$$\begin{aligned}
 \dot{z}_B &= \dot{z}_C + \dot{z}_{B/C} \\
 &= \dot{z}_C + (\omega_{BC} i) \vec{r}_{B/C} \\
 &= \dot{z}_C + (\omega_{BC} i) (\vec{r}_B - \vec{r}_C) \\
 &= \vec{v}_C + (\omega_{BC} i) (0.75 \cos 30 + i(-0.75 \sin 30)) \\
 &= 0.15\omega_{OC} i + (\omega_{BC} i) 0.75 \cos 30 + (\omega_{BC} i) i(-0.75 \sin 30) \\
 &= 0.15\omega_{OC} i + (0.75\omega_{BC} \cos 30) i + i^2 (-0.75\omega_{BC} \sin 30) \\
 &= 0.75\omega_{BC} \sin 30 + (0.15\omega_{OC} + 0.75\omega_{BC} \cos 30) i
 \end{aligned}$$

which can also be written equivalently as

$$\dot{z}_B = 0.75\omega_{BC} \sin 30 + (0.15\omega_{OC} + 0.75\omega_{BC} \cos 30) \hat{k} \times$$

which would give exactly that for vector algebra upon taking components.

With $\dot{z}_B = -18 + 0i$ m/s

$$-18 + 0i = 0.75\omega_{BC} \sin 30 + (0.15\omega_{OC} + 0.75\omega_{BC} \cos 30) i$$

Equating real and imaginary components (which are exactly the same as i -hat and j -hat components)

$$-18 = 0.75\omega_{BC} \sin 30$$

$$0 = 0.15\omega_{OC} + 0.75\omega_{BC} \cos 30$$

which solves the same as that for the vector algebra

$$\begin{aligned}
 \omega_{BC} &= -18 / 0.75 \sin 30 \\
 &= -48 \text{ rad/s} \quad (\text{ClockWise rotation})
 \end{aligned}$$

$$\begin{aligned}
 \omega_{OC} &= 48(0.75 \cos 30 / 0.15) \\
 &= 207.85 \text{ rad/s} \quad (\text{CounterClockWise rotation})
 \end{aligned}$$

The velocity of C on the Argand plane is here ‘purely imaginary’

$$\begin{aligned}\bar{v}_C &= \hat{j}(0.15\omega_{OC}) \\ &= \hat{j}(0.15 \times 207.85) \\ &= \hat{j}(31.2) \text{ m/s}\end{aligned}$$

$$\begin{aligned}\dot{z}_C &= (0.15\omega_{OC})i \\ &= (0.15 \times 207.85)i \quad \text{nb:} \\ &= 0 + 31.2i \text{ m/s}\end{aligned}$$

$$\begin{aligned}\dot{z}_C \hat{i} &= (0 + 31.2\hat{k} \times) \hat{i} \\ &= (\hat{k} \times \hat{i})31.2 \\ &= \hat{j}(31.2) \text{ m/s}\end{aligned}$$

with vector magnitude

$$\begin{aligned}|\bar{v}_C| &= \sqrt{\bar{v}_C' \bar{v}_C} \\ &= \sqrt{|\hat{i}(0)|^2 + |\hat{j}(31.2)|^2} \\ &= \sqrt{0^2 + 31.2^2} \\ &= 31.2 \text{ m/s}\end{aligned}$$

with modulus

$$\begin{aligned}|\dot{z}_C| &= \sqrt{\dot{z}_C' \dot{z}_C} \\ &= \sqrt{(0 - 31.2i)(0 + 31.2i)} \\ &= \sqrt{0^2 + 31.2^2} \\ &= 31.2 \text{ m/s}\end{aligned}$$

and angle

$$\begin{aligned}\angle \bar{v}_C &= \arctan(|\hat{j}(31.2)| / |\hat{i}(0)|) \\ &= \arctan(31.2 / 0) \\ &= \arctan(\infty) \\ &= \pi / 2 \text{ rads} \\ &= 90^\circ\end{aligned}$$

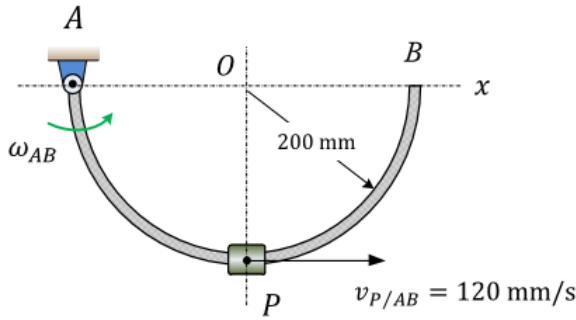
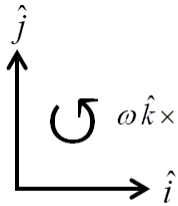
and angle

$$\begin{aligned}\angle \dot{z}_C &= \arctan(\text{Im}(\dot{z}_C) / \text{Re}(\dot{z}_C)) \\ &= \arctan(31.2 / 0) \\ &= \arctan(\infty) \\ &= \pi / 2 \text{ rads} \\ &= 90^\circ\end{aligned}$$

Which give both exactly equivalent steps and results, showing that the results for vector algebra are identical to that of complex algebra with $i = \hat{k} \times$, for all plane kinematics, and demonstrating unequivocally once again that $i = \hat{k} \times$ is the complex number for the xy-plane and, further, that the complex plane (Argand diagram) is identical to the xy-plane for the $\hat{k}$ rotation axis.

Example 2: Relative motion, acceleration (with Coriolis term)

(MMAN2300) The collar P slides from A towards B along a semi-circular rod AB of radius 200 mm. The rod rotates about the pin at A , and the speed of the collar P relative to the rod is constant at 120 mm/s. When the system is in the position shown, the angular velocity of the rod is $\omega_{AB} = 0.8$ rad/s counter clockwise and the angular acceleration of the rod $\alpha_{AB} = 0.5$ rad/s² clockwise. Determine the velocity and acceleration of P at the instant shown.

**Solution by xy plane vector algebra****Data**

$$\omega_{AB} = \omega_{AP} = +0.8 \text{ rad/s}$$

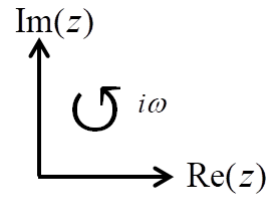
$$\alpha_{AB} = \alpha_{AP} = -0.5 \text{ rad/s}^2$$

$$v_{\text{relative}} = v_{\text{rel}} = 0.12 \text{ m/s}$$

$$a_{\text{relative}} = a_{\text{rel}} = 0 \text{ m/s}^2$$

Velocity vector of point P

$$\begin{aligned} \vec{v}_P &= \vec{v}_{\text{rel}} + \vec{v}_A + \vec{v}_{P/A} \\ &= \vec{v}_{\text{rel}} + \vec{v}_A + \vec{\omega}_{AP} \times \vec{r}_{P/A} \\ &= \vec{v}_{\text{rel}} + \vec{v}_A + \omega_{AB} \hat{k} \times (\vec{r}_P - \vec{r}_A) \\ &= \hat{i}0.12 + \vec{0} + 0.8\hat{k} \times (\hat{i}0.2 - \hat{j}0.2) \\ &= \hat{i}0.12 + 0.8\hat{k} \times (\hat{i}0.2) - 0.8\hat{k} \times (\hat{j}0.2) \\ &= \hat{i}0.12 + (\hat{k} \times \hat{i})0.16 - (\hat{k} \times \hat{j})0.16 \\ &= \hat{i}0.12 + \hat{j}0.16 + \hat{i}0.16 \\ &= \hat{i}0.28 + \hat{j}0.16 \text{ m/s} \end{aligned}$$

Solution by complex plane algebra

$$\omega_{AB} = \omega_{AP} = +0.8 \text{ rad/s}$$

$$\alpha_{AB} = \alpha_{AP} = -0.5 \text{ rad/s}^2$$

$$v_{\text{relative}} = v_{\text{rel}} = 0.12 \text{ m/s}$$

$$a_{\text{relative}} = a_{\text{rel}} = 0 \text{ m/s}^2$$

$$\begin{aligned} \dot{z}_P &= \dot{z}_{\text{rel}} + \dot{z}_A + \dot{z}_{P/A} \\ &= \dot{z}_{\text{rel}} + \dot{z}_A + (\omega_{AP} i) z_{P/A} \\ &= \dot{z}_{\text{rel}} + \dot{z}_A + (\omega_{AP} i)(z_P - z_A) \\ &= (0.12 + 0i) + (0 + 0i) + (0.8i)(0.2 - 0.2i) \\ &= 0.12 + (0.8i)(0.2) - (0.8i)(0.2i) \\ &= 0.12 + 0.16i - 0.16i^2 \\ &= 0.12 + 0.16i + 0.16 \\ &= 0.28 + 0.16i \text{ m/s} \end{aligned}$$

nb: this is $\dot{z}_p = 0.28 + 0.16\hat{k} \times$ which is the same as $\bar{v}_p$ when components are taken (setting the zero phase axis to i -hat)

$$\dot{z}_p \hat{i} = 0.28\hat{i} + 0.16\hat{k} \times \hat{i} = \hat{i}0.28 + \hat{j}0.16 = \bar{v}_p$$

Velocity vector magnitude

$$\begin{aligned} |\bar{v}_p| &= \sqrt{\bar{v}_p' \bar{v}_p} \\ &= \sqrt{(\hat{i}'0.28)(\hat{i}0.28) + (\hat{j}'0.16)(\hat{j}0.16)} \\ &= \sqrt{|\hat{i}0.28|^2 + |\hat{j}0.16|^2} \\ &= \sqrt{0.28^2 + 0.16^2} \\ &= 0.3225 \text{ m/s} \end{aligned}$$

Velocity modulus

$$\begin{aligned} |\dot{z}_p| &= \sqrt{\dot{z}_p' \dot{z}_p} \\ &= \sqrt{(0.28 - 0.16i)(0.28 + 0.16i)} \\ &= \sqrt{0.28^2 + 0.16^2} \\ &= 0.3225 \text{ m/s} \end{aligned}$$

Velocity vector angle (phase)

$$\begin{aligned} \angle \bar{v}_p &= \arctan(|\hat{j}0.16|/|\hat{i}0.28|) \\ &= \arctan(0.16/0.28) \\ &= 0.519 \text{ rad} \\ &= 29.7^\circ \end{aligned}$$

$$\begin{aligned} \angle \dot{z}_p &= \arctan(\text{Im}(\dot{z}_p)/\text{Re}(\dot{z}_p)) \\ &= \arctan(0.16/0.28) \\ &= 0.519 \text{ rad} \\ &= 29.7^\circ \end{aligned}$$

Acceleration vector of point P

$$\begin{aligned} \bar{a}_p &= \bar{a}_{rel} + \bar{a}_{Coriolis} + \bar{a}_A + (\bar{a}_{p/A})_t + (\bar{a}_{p/A})_n \\ &= \bar{a}_{rel} + 2(\bar{\omega}_{AP} \times) \bar{v}_{rel} \\ &\quad + \bar{a}_A + (\bar{\alpha}_{AP} \times) \bar{r}_{pA} + (\bar{\omega}_{AP} \times)^2 \bar{r}_{pA} \\ &= \bar{a}_{rel} + 2\omega_{AP}(\hat{k} \times) \bar{v}_{rel} \\ &\quad + \bar{a}_A + \alpha_{AP}(\hat{k} \times) \bar{r}_{pA} + \omega_{AP}^2(\hat{k} \times)^2 \bar{r}_{pA} \\ &= \bar{a}_{rel} + 2\omega_{AP}(\hat{k} \times) \bar{v}_{rel} \\ &\quad + \bar{a}_A + \alpha_{AP}(\hat{k} \times)(\bar{r}_p - \bar{r}_A) + \omega_{AP}^2(\hat{k} \times)^2(\bar{r}_p - \bar{r}_A) \\ &= \bar{0} + 2(0.8)(\hat{k} \times) \hat{i}0.12 + \bar{0} + ((-0.5)\hat{k} \times) \hat{i}0.20 - \hat{j}0.20 \\ &\quad + ((0.8)^2(\hat{k} \times)^2) \hat{i}0.20 - \hat{j}0.20 \\ &= (\hat{k} \times \hat{i})0.192 + ((\hat{k} \times \hat{i})0.10 + (\hat{k} \times \hat{j})0.10) \\ &\quad + ((-0.8)^2) \hat{i}0.20 - \hat{j}0.20 \\ &= \hat{j}0.192 + ((-\hat{j}0.10 - \hat{i}0.10)) + (-\hat{i}0.128 + \hat{j}0.128) \\ &= \hat{i}(-0.228) + \hat{j}0.220 \text{ m/s}^2 \end{aligned}$$

$$\begin{aligned} \ddot{z}_p &= \ddot{z}_{rel} + \ddot{z}_{Coriolis} + \ddot{z}_A + (\ddot{z}_{pA})_t + (\ddot{z}_{pA})_n \\ &= \ddot{z}_{rel} + 2(\omega_{AP} \hat{i}) \dot{z}_{rel} \\ &\quad + \ddot{z}_A + (\alpha_{AP} \hat{i}) z_{pA} + (\omega_{AP} \hat{i})^2 z_{pA} \\ &= \ddot{z}_{rel} + 2(0.8i) \dot{z}_{rel} \\ &\quad + \ddot{z}_A + (0.5i) z_{pA} + (0.8^2 i^2) z_{pA} \\ &= (0 + 0i) + 2(0.8i)(0.12 + 0i) \\ &\quad + (0 + 0i) + (-0.5i)(0.2 - 0.2i) - 0.64(0.2 - 0.2i) \\ &= 2(0.8i)(0.12) \\ &\quad + (-0.5i)(0.2 - 0.2i) - 0.64(0.2 - 0.2i) \\ &= 0.192i + (-0.100i + 0.100i^2) - 0.128 + 0.128i \\ &= -0.228 + 0.220i \text{ m/s}^2 \end{aligned}$$

nb: this is equivalent to $\ddot{z}_p = -0.228 + 0.220\hat{k} \times$, i.e.,

$$\ddot{z}_p \hat{i} = -0.228\hat{i} + 0.220\hat{k} \times \hat{i} = \hat{i}(-0.228) + \hat{j}0.220 = \bar{a}_p$$

Acceleration vector magnitude

$$\begin{aligned}
|\vec{a}_p| &= \sqrt{\vec{a}'_p \vec{a}_p} \\
&= \sqrt{(-\hat{i}'0.228)(-\hat{i}'0.228) + (\hat{j}'0.220)(\hat{j}'0.220)} \\
&= \sqrt{|-\hat{i}'0.228|^2 + |\hat{j}'0.220|^2} \\
&= \sqrt{0.228^2 + 0.220^2} \\
&= 0.3168 \text{ m/s}^2
\end{aligned}$$

Acceleration modulus

$$\begin{aligned}
|\ddot{z}_p| &= \sqrt{\ddot{z}'_p \ddot{z}_p} \\
&= \sqrt{(-0.228 - 0.228i)(-0.228 + 0.228i)} \\
&= \sqrt{0.228^2 + 0.220^2} \\
&= 0.3168 \text{ m/s}^2
\end{aligned}$$

Acceleration vector angle (phase)

$$\begin{aligned}
\angle \vec{a}_p &= \pi - \arctan(|\hat{j}'0.220| / |\hat{i}'0.228|) \\
&= \pi - \arctan(0.220 / 0.228) \\
&= \pi - 0.768 \\
&= 2.374 \text{ rad} \\
&= 136.0^\circ
\end{aligned}$$

$$\begin{aligned}
\angle \ddot{z}_p &= \pi + \arctan(\text{Im}(\ddot{z}_p) / \text{Re}(\ddot{z}_p)) \\
&= \pi - \arctan(0.220 / 0.228) \\
&= \pi - 0.768 \\
&= 2.374 \text{ rad} \\
&= 136.0^\circ
\end{aligned}$$

It all works perfectly, with $i = \hat{k} \times$. This was an easier acceleration problem but it all works out perfectly with the difficult ones as well (where the angular rotational accelerations have to be found by solving simultaneous equations), right up to the most difficult (and intricate) of the difficult.

To demonstrate that the operation of the vector magnitude is equivalent to that of the taking the complex modulus, the acceleration vector is

$$\vec{a}_p = \hat{i}(-0.228) + \hat{j}0.220 = (-0.228 + 0.220\hat{k} \times)\hat{i}$$

Inner product for obtaining the squared vector magnitude $\vec{a}'_p \vec{a}_p$, here equivalent to taking the complex modulus with the exact same steps,

$$\begin{aligned}
\vec{a}'_p \vec{a}_p &= ((-0.228 + 0.220\hat{k} \times)\hat{i})'(-0.228 + 0.220\hat{k} \times)\hat{i} \\
&= \hat{i}'(-0.228 + 0.220\hat{k} \times)'(-0.228 + 0.220\hat{k} \times)\hat{i} \\
&= \hat{i}'(-0.228 + 0.220(\hat{k} \times))'(-0.228 + 0.220\hat{k} \times)\hat{i} \\
&= \hat{i}'(-0.228 + 0.220(-\hat{k} \times))'(-0.228 + 0.220\hat{k} \times)\hat{i} \\
&= \hat{i}'(0.228^2 + 0.220^2)\hat{i} \\
&= \hat{i}'\hat{i}(0.228^2 + 0.220^2) \\
&= 0.228^2 + 0.220^2
\end{aligned}$$

$$\sqrt{\vec{a}'_p \vec{a}_p} = \sqrt{0.228^2 + 0.220^2} = 0.3168$$

Exactly as for complex modulus, same thing with $i = \hat{k} \times$. Also, to prove further the equivalence of vector magnitudes and complex modulus, with diadically expanded operators into matrix bases

$$\bar{a}_p = \hat{i}(-0.228) + \hat{j}0.220 = (-0.228 + 0.220\hat{k} \times)\hat{i} = \left(-0.228I + 0.220 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.228 & -0.220 & 0 \\ 0.220 & -0.228 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which has transpose (conjugate)

$$\bar{a}'_p = (\hat{i}(-0.228) + \hat{j}0.220)' = \hat{i}'(-0.228 + 0.220\hat{k} \times)' = \left(\begin{pmatrix} -0.228 & -0.220 & 0 \\ 0.220 & -0.228 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)' = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -0.228 & 0.220 & 0 \\ -0.220 & -0.228 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore

$$\bar{a}'_p \bar{a}_p = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -0.228 & 0.220 & 0 \\ -0.220 & -0.228 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -0.228 & -0.220 & 0 \\ 0.220 & -0.228 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.228 & 0.220 & 0 \end{pmatrix} \begin{pmatrix} -0.228 \\ 0.220 \\ 0 \end{pmatrix} = 0.228^2 + 0.220^2$$

$$\sqrt{\bar{a}'_p \bar{a}_p} = \sqrt{0.228^2 + 0.220^2} = 0.3168$$

So the conjugate works exactly the same as for complex numbers (here for relative motion kinematics), here proved by the diadically expanded matrix bases for the vector plane. The conjugate of the complex number is equivalent to the transpose of $\hat{k} \times$, demonstrated here once again. It is all equivalent with $i = \hat{k} \times$. Proved here again definitively that $i = \hat{k} \times$ is the complex number (basis rotation operator for a single orthogonal plane of R^3).

That is all very fine and well, but where was the rotation eigenvalue and eigenvector when obtaining the vector magnitude? Should it be included? These kinematic problems are a static snapshot of the dynamics at a specific point in time, essentially solved for the differential dynamics. When integrated into integral form, the integration process adds the $\hat{k}$ rotation eigenvalue and eigenvector as for standard eigenvalue problems working in an exponential base. The $\hat{k}$ eigenvalue is zero for the magnitude/modulus on purpose, as it is the singular complex number from which the modulus is being evaluated. So no, it is right not to include it in these sorts of problems (it should be zero with the singular, rather than full rank, complex number), those of static snapshots in time of kinematic differential forms. The solutions to these sorts of kinematics problems then provide initial conditions, or intermediate points, for forward dynamics in harmonic motions (harmonic because of $i = \hat{k} \times \dots$ notice how the problems all involve circles?).

I also mentioned that the majority of acceleration problems are more difficult than Example 2. As briefly mentioned, this is because unknown scalar angular accelerations have to be obtained by solving two large simultaneous equations. This can be made much easier by transforming to normal-tangential coordinates, which is the same mathematics as a change of base transformation to polar cylindricals. It is also equivalent to a rotation onto principal axes in an eigenvalue problem. It is also

exactly equivalent to a rotation onto the principal axes of the Argand diagram with the complex exponential, exactly with $i = \hat{k} \times$. So I'll just state the formulae here again, which proved useful to me whilst solving the relative motion kinematic problems whilst sitting engineering mechanics. I'll also show the equivalent complex algebra which is exactly the same process with different notation (note also that normal and tangential coordinates are the same as polar cylindrical radius and angle so t -hat and n -hat are alternative notation for r -hat and θ -hat... here I have chosen t -hat and n -hat because normal-tangential coordinates are the standard convention for engineering mechanics):

$$\begin{aligned} \hat{t} &= \exp(\hat{\theta} \times) \hat{i} & \text{and} & & \text{Re}(z)^* &= \exp(\theta i) \text{Re}(z) \\ \hat{n} &= \exp(\hat{\theta} \times) \hat{j} & & & i \text{Im}(z)^* &= \exp(\theta) i \text{Im}(z) \end{aligned} \quad \text{are exactly equivalent}$$

the t -hat (r -hat) component being the rotated i -hat component for the x -axis (or real axis in the complex plane) and the n -hat (θ -hat) component being the rotated j -hat component for the y -axis,

$$\begin{aligned} \hat{t} &= \exp(\hat{\theta} \times) \hat{i} = \exp(\theta) \hat{i} = (\cos \theta + (\sin \theta) \hat{k} \times) \hat{i} = (\cos \theta) \hat{i} + (\sin \theta) \hat{k} \times \hat{i} = (\cos \theta) \hat{i} + (\sin \theta) \hat{j} = \hat{i} \cos \theta + \hat{j} \sin \theta \\ \hat{n} &= \exp(\hat{\theta} \times) \hat{j} = \exp(\theta) \hat{j} = (\cos \theta + (\sin \theta) \hat{k} \times) \hat{j} = (\cos \theta) \hat{j} + (\sin \theta) \hat{k} \times \hat{j} = (\cos \theta) \hat{j} - (\sin \theta) \hat{i} = -\hat{i} \sin \theta + \hat{j} \cos \theta \end{aligned}$$

i.e.,

$$\begin{aligned} \hat{t} &= \hat{i} \cos \theta + \hat{j} \sin \theta & \text{Re}(z)^* &= \exp(\theta i) \text{Re}(z) = e^{\theta i} = \cos \theta + i \sin \theta \\ \hat{n} &= -\hat{i} \sin \theta + \hat{j} \cos \theta & i \text{Im}(z)^* &= \exp(\theta) i \text{Im}(z) = e^{\theta} i = (\cos \theta + i \sin \theta) i = -\sin \theta + i \cos \theta \end{aligned}$$

which rotates the differential system onto the principal axes so that the unknown quantities are on the $\hat{i}\hat{i}$ and $\hat{j}\hat{j}$ principal axes (i.e., diagonalized) and do not have to be solved simultaneously from equations arising from the $\hat{i}\hat{j}$ and $\hat{j}\hat{i}$ components of the matrix basis of the vector plane.

Once the problem is solved in tangential-normal (polar cylindrical) coordinates, the vector magnitudes are the same as the original coordinates and the vector angle adjusted accordingly by the rotation angle. Alternatively, or to give the full vector solution in the original coordinates, the change of base back to i -hat and j -hat is then

$$\begin{aligned} \hat{i} &= \exp(-\hat{\theta} \times) \hat{t} & \text{and} & & \text{Re}(z) &= \exp(-\theta i) \text{Re}(z)^* \\ \hat{j} &= \exp(-\hat{\theta} \times) \hat{n} & & & i \text{Im}(z) &= \exp(-\theta) i \text{Im}(z)^* \end{aligned}$$

which is equivalent to inverting back to the original basis by inverting back the rotation matrix [which is full rank although with the rotation eigenvector and eigenvalue latent]

$$\begin{aligned} \hat{i} &= \exp(-\hat{\theta} \times) \hat{t} = \exp(-\theta) \hat{t} = (\cos \theta - (\sin \theta) \hat{k} \times) \hat{t} = (\cos \theta) \hat{t} - (\sin \theta) \hat{k} \times \hat{t} = (\cos \theta) \hat{t} - (\sin \theta) \hat{n} = \hat{t} \cos \theta - \hat{n} \sin \theta \\ \hat{j} &= \exp(-\hat{\theta} \times) \hat{n} = \exp(-\theta) \hat{n} = (\cos \theta - (\sin \theta) \hat{k} \times) \hat{n} = (\cos \theta) \hat{n} - (\sin \theta) \hat{k} \times \hat{n} = (\cos \theta) \hat{n} + (\sin \theta) \hat{t} = \hat{t} \sin \theta + \hat{n} \cos \theta \end{aligned}$$

i.e.,

$$\hat{i} = \hat{t} \cos \theta - \hat{n} \sin \theta$$

$$\hat{j} = \hat{t} \sin \theta + \hat{n} \cos \theta$$

This method, change of base with **expk**, all works out marvelously for engineering mechanics, with for example, ‘Motion Relative to a Rotating Axis’ problems of Miriam and Kraige [I used this method in a couple of assignments and, as an ongoing theme with the complex number $i = \hat{k} \times$ documents I have been writing, it all worked out like a dream].

The change of base rotation onto the principal axes with **expk** has hence been shown to be exactly equivalent, one for one, with the complex exponential with $i = \hat{k} \times$ given that it ($e^{i\theta}$) is equivalent to the rotation matrix (**expk**) for the $\hat{k}$ rotation axis (as I have proved in other documents). All complex analysis can now be placed in a real vector space, and this knowledge now allows for greater kinematic analysis in full R^3 volumes with the additional rotation operators of $\mathbf{j} \times$ and $\mathbf{i} \times$ and **expj** and **expi** (hopefully extending Euler dynamics).

Theorem: The R^3 matrix basis is the R^3 dyadic cubic tensor.

References

Miriam, J.L. and Kraige, L.G., **Engineering Mechanics Vol. 2 DYNAMICS 2nd Ed.** John Wiley and Sons. New York.

MMAN2300 Engineering Mechanics, **Course Notes** (relative motion example problems), UNSW Kensington, Sydney.